

On principles of large deviation and selected data compression

Y. Suhov¹, I. Stuhl²

Abstract

The Shannon Noiseless coding theorem (the data-compression principle) asserts that for an information source with an alphabet $\mathcal{X} = \{0, \dots, \ell - 1\}$ and an asymptotic equipartition property, one can reduce the number of stored strings $(x_0, \dots, x_{n-1}) \in \mathcal{X}^n$ to ℓ^{nh} with an arbitrary small error-probability. Here h is the entropy rate of the source (calculated to the base ℓ). We consider further reduction based on the concept of utility of a string measured in terms of a rate of a weight function. The novelty of the work is that the distribution of memory is analyzed from a probabilistic point of view. A convenient tool for assessing the degree of reduction is a probabilistic large deviation principle. Assuming a Markov-type setting, we discuss some relevant formulas, including the case of a general alphabet.

1 Introduction

Consider a discrete-time random process $\mathbf{X} = (X_n)$, $n \in \mathbb{Z}_+ := \{0, 1, 2, \dots\}$, where the random variable X_n – possibly a random vector or a random element in a space \mathcal{X} – describes the state of the process at time n . One interpretation used throughout the paper is that process \mathbf{X} represents an *information source*, in the spirit of [1], [2]; here set \mathcal{X} will play role of a source alphabet. Under such an interpretation the probability distribution of \mathbf{X} (on $\mathcal{X}^{\mathbb{Z}_+}$) is referred to as \mathbb{P}^{so} . Sample states of the process are given by points $x \in \mathcal{X}$. An (initial) n -string is a collection $\mathbf{x}_0^{n-1} = \{x_i : 0 \leq i < n\} \in \mathcal{X}^n$; n is referred to as the length of \mathbf{x}_0^{n-1} . A random sample drawn from \mathbf{X} is denoted by \mathbf{X}_0^{n-1} ; it is a random element in \mathcal{X}^n . The probability distribution for \mathbf{X}_0^{n-1} generated by \mathbb{P}^{so} is denoted by p_n^{so} (i.e., $\mathbb{P}^{\text{so}}(\mathbf{X}_0^{n-1} \in \mathcal{B}_n) = p_n^{\text{so}}(\mathcal{B}_n)$, for any (Borel) set $\mathcal{B}_n \subseteq \mathcal{X}^n$). For a process with discrete states (with a finite or countable alphabet \mathcal{X}), the value $p_n^{\text{so}}(\mathbf{x}_0^{n-1}) = \mathbb{P}^{\text{so}}(\mathbf{X}_0^{n-1} = \mathbf{x}_0^{n-1})$. In this context, the concepts of information and entropy rates are relevant; see below.

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¹ Mathematics Dept., Penn State University, University Park, State College, PA 16802, USA; DPMMS, University of Cambridge, UK;

E-mail: yms@statslab.cam.ac.uk

² Mathematics Dept., University of Denver, Denver, CO 80208 USA; Appl. Math and Prob. Theory Dept., University of Debrecen, Debrecen, 4028, HUN;

E-mail: izabella.stuhl@du.edu

However, there are situations where one may need to extend (or complement) standard notions. In this work we are motivated by Refs [3, 4] discussing *weighted* information and entropy. These concepts emerge when one introduces a *weight function* $\phi_n(\mathbf{x}_0^{n-1})$ reflecting *utility* of an outcome string \mathbf{x}_0^{n-1} .

A second interpretation emerges when we consider the problem of *storing* strings \mathbf{x}_0^{n-1} . Suppose we have a notion of ‘volume’ in \mathcal{X} associated with a measure ν with $V = \nu(\mathcal{X}) < \infty$ (e.g., the number of points in a set $\mathcal{A} \subseteq \mathcal{X}$ in the case of a finite alphabet). Then the volume in \mathcal{X}^n may be represented by the product-measure ν^n . A normalized volume $\frac{\nu^n(\mathcal{B}_n)}{V^n}$, $\mathcal{B}_n \subseteq \mathcal{X}^n$, gives a probability distribution on \mathcal{X}^n (with IID digits), and an (asymptotic) analysis of ν^n is reduced to an analysis of this probability distribution. When the cardinality $\#(\mathcal{X}) = \ell$ is finite and $\nu(\mathcal{A}) = \#\mathcal{A}$ (a counting measure), we obtain $V = \ell$. The volume of a set $\mathcal{B}_n \subseteq \mathcal{X}^n$ is written as $\#\mathcal{B}_n = \ell^n p_n^{\text{eq}}(\mathcal{B}_n)$ where p_n^{eq} stands for an equidistribution on \mathcal{X}^n , with $p_n^{\text{eq}}(\mathbf{x}_0^{n-1}) = 1/\ell^n$ for all $\mathbf{x}_0^{n-1} \in \mathcal{X}^n$.

More generally, we can think of a probability distribution p_n^{st} on \mathcal{X}^n such that the volume in \mathcal{X}^n is represented by $V_n p_n^{\text{st}}(\mathcal{B}_n)$, $\mathcal{B}_n \subseteq \mathcal{X}^n$, where V_n is a given constant (yielding the total amount of memory (or space in a broader sense) available for storing strings of length n). Then asymptotic properties of p_n^{st} can be used for assessing the volume of random strings \mathbf{X}_0^{n-1} generated by p_n^{so} . In this paper, such an approach is used for the purpose of *selected data compression*.

Returning to the information source interpretation, the standard (Shannon) information $I(\mathbf{x}_0^{n-1})$ and entropy $H(p_n^{\text{so}})$ of the source n -string is given by

$$I(\mathbf{x}_0^{n-1}) = -\log p_n^{\text{so}}(\mathbf{x}_0^{n-1}), \quad H(p_n^{\text{so}}) = \sum_{\mathbf{x}_0^{n-1} \in \mathcal{X}^n} p_n^{\text{so}}(\mathbf{x}_0^{n-1}) I(\mathbf{x}_0^{n-1}). \quad (1.1)$$

The rates

$$i = \lim_{n \rightarrow \infty} \frac{I(\mathbf{X}_0^{n-1})}{n} \quad \mathbb{P}^{\text{so}}\text{-a.s.}, \quad \text{and} \quad h = \lim_{n \rightarrow \infty} \frac{H(p_n^{\text{so}})}{n} \quad (1.2)$$

are fundamental parameters of a random process leading to profound results and fruitful theories with far-reaching consequences, cf. [1, 2].¹⁾ In fact, under mild assumptions, $h = i$: this is the Shannon–McMillan–Breiman theorem [1, 7].

In this paper, we treat two types of weight functions $\phi_n(\mathbf{x}_0^{n-1})$: additive and multiplicative; see below. A justification of our approach can be provided through aforementioned selected data compression. The basic idea of the Shannon Noiseless coding theorem (NCT), or data-compression (DC), was to disregard strings/messages \mathbf{x}_0^{n-1} of length $n \gg 1$ (drawn from p_n^{so}) which are highly unlikely. (That is, with low probabilities $p_n^{\text{so}}(\mathbf{x}_0^{n-1})$, or, equivalently, with high information $I(\mathbf{x}_0^{n-1})$, for discrete outcomes.) Incidentally, one also disregards strings that are highly likely. The remaining strings, forming set $\mathcal{T}_n \subset \mathcal{X}^n$ with $p_n^{\text{so}}(\mathcal{T}_n) \rightarrow 1$, can be characterized through the information/entropy rate (IER) $h = i$ by invoking the asymptotic equipartition property (AEP). Pictorially, all strings $\mathbf{x}_0^{n-1} \in \mathcal{T}_n$ carry, approximately, the same IER $i = h$; cf. (1.2). Assume, until a further note, that the total number of n -strings equals ℓ^n where $\#\mathcal{X} = \ell < \infty$. Then the DC allows us to diminish the amount of memory needed to store strings $\mathbf{x}_0^{n-1} \in \mathcal{T}_n$

¹⁾As a rule (with exceptions), references of a general character are given to books rather than to original papers.

by reducing their length from n to $\frac{nh}{\log \ell}$. (Such a reduction is effectuated by a lossless coding.) Here $h \leq \log \ell$ (and in many realistic situations, $h \ll \log \ell$). The probability $p_n^{\text{so}}(\mathcal{X}^n \setminus \mathcal{T}_n)$ of information loss is kept small because of the AEP (which is a Law of large numbers for $I(\mathbf{X}_0^{n-1})$.)

Now, one may be interested in a further reduction of the used memory by extracting and storing only ‘valuable’ strings. It can be done by using a given weight function (WF) ϕ_n : strings \mathbf{x}_0^{n-1} with high growth rates of $\phi_n(\mathbf{x}_0^{n-1})$ are stored while others disregarded.

We show that selecting most valued strings yields a further reduction of the storage memory, and its effect can be estimated numerically. The number of strings in a selected set $\mathcal{B}_n \subseteq \mathcal{T}_n$ is given by $\#\mathcal{B}_n = \ell^n p_n^{\text{eq}}(\mathcal{B}_n)$. Hence,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \#\mathcal{B}_n = \gamma \quad \text{where} \quad \gamma := \log \ell + \kappa, \quad \kappa := \lim_{n \rightarrow \infty} \frac{1}{n} \log p_n^{\text{eq}}(\mathcal{B}_n). \quad (1.3)$$

More generally, when we measure the volume of \mathcal{B}_n by $V_n p_n^{\text{st}}(\mathcal{B}_n)$, we encounter the limit $\gamma = v + \kappa$. Here $v = \lim_{n \rightarrow \infty} \frac{1}{n} \log V_n$ and

$$\kappa := \lim_{n \rightarrow \infty} \frac{1}{n} \log p_n^{\text{st}}(\mathcal{B}_n). \quad (1.4)$$

Passing from p_n^{eq} to p_n^{st} makes the amount of memory needed to store \mathbf{x}_0^{n-1} string-dependent and separate the issues of the total volume V_n and that of the distribution of the storage volume between different strings.

The value κ in (1.4) can be studied via the Large deviation (LD) theory. The LD studies are now an established trend in theoretical and applied probability; some important reference sources are [10]–[20]. The core is the LD principle (LDP); its gist (as we use it in this work) is summarized in the formula

$$\kappa = -\inf \left[\Pi^*(z) : z \in B \right]. \quad (1.5)$$

Here B is a set of probabilistic vectors (empirical measures) in a (suitable) Euclidean space (\mathbb{R}^ℓ or \mathbb{R}^{ℓ^2} and so on); the form of B depends upon the choice of sets \mathcal{B}_n . (Set B is constructed from a frequency analysis of strings $\mathbf{x}_0^{n-1} \in \mathcal{B}_n$ and turns out to be a convex polyhedron.) Next, Π^* is a large deviation rate (LDR) function. Typically, Π^* is a lower semi-continuous convex function representing the Legendre–Fenchel transform of some moment-generating function Π (this fact is encapsulated in the Gärtner–Ellis theorem). Furthermore, Π^* has a form of a relative entropy (an observation going back to the 1957 Sanov theorem; cf. e.g., [1] and [10]–[20]). Next, $\Pi^*(z) = 0$ at a point (or points) z representing a related expected value and $\Pi^*(z) > 0$ at all other points z (this includes values $\Pi^*(z) = +\infty$). It yields that $\gamma \leq \log \ell$, and in many cases $\gamma \ll \log \ell$, depending on the choice of \mathcal{B}_n . Consequently, the reduction in length is from n to $\frac{n\gamma}{\log \ell}$. (In our situation, as $\mathcal{B}_n \subset \mathcal{T}_n$, value γ will be $< h$, achieving a distinct improvement compared with the Shannon NCT.) It is helpful that both B and Π^* admit some ‘standard’ representations reflecting the structure of p_n^{so} (and ϕ_n) in a natural (and computationally convenient) manner allowing to calculate the value γ . This is the thrust

of our approach: the LDP is used for p_n^{st} whereas p_n^{so} and ϕ_n specify B and Π^* (through selected sets \mathcal{B}_n).

A basic condition adopted in this paper is that each of distributions p_n^{so} and p_n^{st} is generated by a discrete-time Markov chain (DTMC). Although the LDP scheme is formally applicable in a more general situation, the Markov assumption will allow us to simplify technicalities. For additive and multiplicative WFs ϕ_n we write down formulas for the value κ in (1.4) and (1.5) and specify them when $p_n^{\text{st}} = p_n^{\text{eq}}$. As was said above, such an approach allows for varying both the set \mathcal{B}_n (that is, threshold values for utility and information carried by a selected string) and the distribution p_n^{st} of the normalized volume allocated to different strings. In practical terms, it means that one can predict an impact of an adaptation of storage principles to changing demands and conditions.

In Sect 2 we deal with the case of a finite alphabet with $\#\mathcal{X} = \ell$, cf. (2.5), (2.8), (2.9), (2.11); Sect 3 treats a general case, cf. (3.6), (3.9), (3.12). As was said, we focus upon two kinds of WFs: (a) additive and (b) multiplicative. In the simplest form:

$$(a) \phi_n(\mathbf{x}_0^{n-1}) = \sum_{i=0}^{n-1} \varphi_1(x_i), \quad (b) \phi_n(\mathbf{x}_0^{n-1}) = \prod_{i=0}^{n-1} \psi_1(x_i). \quad (1.6)$$

Here $x \in \mathcal{X} \mapsto \varphi_1(x)$ and $x \in \mathcal{X} \mapsto \psi_1(x)$ are given functions (one-digit WFs); for brevity, we write $\varphi(x)$ and $\psi(x)$. Additive WFs may emerge in relatively stable situations where each observed digit X_i brings reward or loss $\varphi(X_i)$; the utility value $\phi_n(\mathbf{X}_0^{n-1})$ is treated as a cumulative gain or deficit after n trials. Multiplicative WFs reflect a more turbulent scenario where the value (e.g., a fortune; see [5]) increases/decreases by a factor $\psi(X_i)$ when outcome X_i is observed.

The topic of this work is closely related to the topic of weighted information/weighted entropy rates; see [6].

N.B. In this work we do **not** claim new LD results, offering instead some new prospects of the LD methodology.

2 Selected data-compression for a finite-alphabet Markov source

2.1

We start with additive one-digit WFs $\phi_n(\mathbf{x}) = \sum \varphi(x_i)$, $\mathbf{x} = (x_0, \dots, x_{n-1})$; cf. (1.6). Assume that probabilities p_n^{st} are generated by an irreducible and aperiodic DTMC where the state space $\mathcal{X} = \{0, \dots, \ell-1\}$. Let $\mathbf{P}^{\text{st}} = (\mathbf{p}_{ij}^{\text{st}})$ and $\lambda = (\lambda(j))$ designate the transition matrix (TM) and an initial distribution. Then $p_n^{\text{st}}(\mathbf{x}) = \lambda(x_0) \prod \mathbf{p}_{x_i x_{i+1}}^{\text{st}}$. We will analyze occupancy-fraction vectors $\mathbf{U}^{(n)} = (U_i^{(n)})$ and $\mathbf{T}^{(n)} = (T_{ij}^{(n)})$, of dimensions ℓ and ℓ^2 . For

$i, j \in \mathcal{X}$, entries $U_j^{(n)} = U_j^{(n)}(\mathbf{x})$ and $T_{j,j'}^{(n)} = T_{j,j'}^{(n)}(\mathbf{x})$ are given by

$$U_i^{(n)} = \frac{1}{n} \sum_{l=0}^{n-1} \mathbf{1}_{x_l=i}, \quad T_{i,j}^{(n)} = \frac{1}{n-1} \sum_{l=0}^{n-2} \mathbf{1}_{x_l=i, x_{l+1}=j},$$

with $\phi_n(\mathbf{x}) = n \sum_{i \in \mathcal{X}} U_i^{(n)} \varphi(i)$ and

$$\log p_n^{\text{st}}(\mathbf{x}) = \log \lambda(x_0) + (n-1) \sum_{i,j \in \mathcal{X}} T_{ij}^{(n)} \log \mathbf{p}_{ij}^{\text{st}}. \quad (2.1)$$

Assume in addition that the source probabilities p_n^{so} are also generated by an irreducible and aperiodic DTMC with a TM $\mathbf{P}^{\text{so}} = (\mathbf{p}_{ij}^{\text{so}})$ and equilibrium distribution $\pi^{\text{so}} = (\pi_i^{\text{so}})$. The IER h in (1.2) takes the form $h = - \sum_{i,j} \pi_i^{\text{so}} \mathbf{p}_{ij}^{\text{so}} \log \mathbf{p}_{ij}^{\text{so}}$.

Given $\epsilon, \eta > 0$, the selected set $\mathcal{B}_n = \mathcal{B}_n(\epsilon, \eta) \subseteq \mathcal{X}^n$ is

$$\mathcal{B}_n = \left\{ \mathbf{x} : \sum_i U_i^{(n)} \varphi(i) \geq \eta, - \sum_{i,j} T_{ij}^{(n)} \log \mathbf{p}_{ij}^{\text{so}} \leq h + \epsilon \right\}. \quad (2.2)$$

We choose \mathcal{B}_n to be a subset in $\mathcal{T}_n = \{\mathbf{x} : T_{ij}^{(n)} \log \mathbf{p}_{ij}^{\text{so}} \leq h + \epsilon\}$, the set which has probability $p_n^{\text{so}}(\mathcal{T}_n) \rightarrow 1$ but it is not required by our method.

We will use the LDP under p_n^{st} (in our case for vectors $\mathbf{U}^{(n)}$ and $\mathbf{T}^{(n)}$) to assess the volume of \mathcal{B}_n . The corresponding LDR functions are denoted by $M^*(y)$ and $\Pi^*(z)$; they are specified below, in (2.3). (Function $\Pi^*(z)$ can be considered as a natural ‘extension’ of $M^*(y)$.) The form of the LDP is standard, and we do not write it in detail for the sake of economy of space.

2.2

Consider the $(\ell-1)$ - and (ℓ^2-1) -dimensional *simplexes* of probability vectors $y = (y_i) \in \mathbb{R}^\ell$ and $z = (z_{ij}) \in \mathbb{R}^{\ell^2}$, respectively:

$$\mathbb{S}_\ell = \{y = (y_i) : y_i \geq 0, \sum y_i = 1\},$$

$$\mathbb{S}_{\ell^2} = \{z = (z_{ij}) : z_{ij} \geq 0, \sum z_{ij} = 1\}.$$

Then $M^*(y) = \infty$ for $y \in \mathbb{R}^\ell \setminus \mathbb{S}_\ell$ and $\Pi^*(z) = \infty$ for $z \in \mathbb{R}^{\ell^2} \setminus \mathbb{S}_{\ell^2}$. Given $y = (y_j) \in \mathbb{S}_\ell$, $z = (z_{ij}) \in \mathbb{S}_{\ell^2}$ and $u = (u_l) \in \mathbb{S}_\ell$, set:

$$M^*(y) = \sup_{u \in \mathbb{S}_\ell} \left[\sum_j y_j \log \frac{u_j}{(\mathbf{P}^{\text{st}} u)_j} \right]$$

$$\Pi^*(z) = \sup_{u \in \mathbb{S}_\ell} \left[\sum_{i,j} z_{ij} \log \frac{u_j}{(\mathbf{P}^{\text{st}} u)_j} \right] \quad \text{where } (\mathbf{P}^{\text{st}} u)_j = \sum_l \mathbf{p}_{jl}^{\text{st}} u_l. \quad (2.3)$$

Functions M^* and Π^* are determined by TM \mathbf{P}^{st} : $M^*(y) = M^*(\mathbf{P}^{\text{st}}; y)$ and $\Pi^*(z) = \Pi^*(\mathbf{P}^{\text{st}}; z)$. This is a standard form of an LDR function for occupancies in a Markov case (which holds in a more general situation). Cf. [11], Ch. 4.1, particularly Lemmas 4.1.36 and 4.1.40, Theorem 4.1.43 and Lemma 4.1.45, and [10], Ch. 6.5, especially Theorems 6.5.2 and 6.5.4. A simple explicit formula for $M^*(y)$ when $\ell = 2$ was proposed in [8]. See also Sect 2.4 below.

Define a convex polyhedron $B = B(\mathbf{P}^{\text{so}}, \epsilon, \eta) \subseteq \mathbb{S}_{\ell^2}$:

$$B = \left\{ z : -\sum_{i,j} z_{ij} \log \mathbf{p}_{ij}^{\text{so}} \leq h + \epsilon, \sum_{i,j} z_{ij} \varphi(i) \geq \eta \right\}. \quad (2.4)$$

Applying general LD results yields

Theorem 2.1 *For all $\epsilon, \eta > 0$ and initial distribution λ , the following relation holds for $\kappa = \kappa(\mathbf{P}^{\text{eq}}, \mathbf{P}^{\text{so}}, \epsilon, \eta)$:*

$$\kappa := \lim_{n \rightarrow \infty} \frac{1}{n} \log p_n^{\text{st}}(\mathcal{B}_n) = -\inf \left[\Pi^*(z) : z \in B \right]. \quad (2.5)$$

Here \mathcal{B}_n , Π^* and B are as in (2.2) – (2.4).

Further, suppose TM \mathbf{P}^{st} has entries of the form $\mathbf{p}_{ij}^{\text{st}} = \mathbf{p}_j$ where vector $\mathbf{p} = (\mathbf{p}_j) \in \mathbb{S}_{\ell}$. Then $\mathbf{P}^{\text{st}} u = \mathbf{p}$ for all $u \in \mathbb{S}_{\ell}$, and (2.3) for $M^*(y)$ features the relative entropy $D(y||\mathbf{p}) = \sum y_j \log \frac{y_j}{\mathbf{p}_j}$. Namely,

$$M^*(y) = \sup \left[\sum y_j \log \frac{u_j}{\mathbf{p}_j} : u = (u_j) \in \mathbb{S}_{\ell} \right] = \sum y_j \log \frac{y_j}{\mathbf{p}_j} \quad (2.6)$$

whenever $y = (y_j) \in \mathbb{S}_{\ell}$, in agreement with the Sanov theorem. Thus, the value $M^*(y)$ in (2.3) can be considered as an analog of relative entropy $D(y||\mathbf{p})$ where vector \mathbf{p} is replaced by \mathbf{P}^{st} , a stochastic TM.

For $p_n^{\text{st}} = p_n^{\text{eq}}$, vector $\mathbf{p} = \mathbf{p}^{\text{eq}} = (1/\ell, \dots, 1/\ell)$, and

$$M^*(y) = D(y||\mathbf{p}^{\text{eq}}) = \log \ell - H(y) \text{ where} \quad (2.7)$$

$$H(y) = -\sum y_i \log y_i \text{ is the entropy of } y = (y_j).$$

In this case, (2.5) yields (see also (1.3)):

$$\kappa = \kappa(\mathbf{P}^{\text{eq}}, \mathbf{P}^{\text{so}}, \epsilon, \eta) = -\log \ell + \gamma. \quad (2.8)$$

Here $\gamma = \gamma(\mathbf{P}^{\text{so}}, \epsilon, \eta)$ is a supremum on the set $A = A(\mathbf{P}^{\text{so}}, \epsilon, \eta) \subset \mathbb{S}_{\ell}$:

$$\gamma = \sup \left[H(y) : y \in A \right] \quad (2.9)$$

where

$$A = \left\{ y = (y_i) : \sum y_i \varphi(i) \geq \eta \text{ and } \exists \text{ a vector } z = (z_{ij}) \in \mathbb{R}_+^{\ell} \right. \\ \left. \text{with } \sum_j z_{ij} = y_i \ \forall \ i \in \mathcal{X} \text{ and } -\sum_{i,j} z_{ij} \log \mathbf{p}_{ij}^{\text{so}} \leq h + \epsilon \right\} \quad (2.10)$$

and $H(y)$ is as in (2.7). Again observe that $A \subseteq \mathbb{S}_{\ell}$ is a convex polyhedron. Since $y \in \mathbb{S}_{\ell} \mapsto H(y)$ is a (strictly) concave function, we have a dichotomy. Either point $(1/\ell, \dots, 1/\ell) \in A$ in which case $\gamma(\mathbf{P}^{\text{so}}, \epsilon, \eta) = \log \ell$ or else $(1/\ell, \dots, 1/\ell) \notin A$, $\gamma(\mathbf{P}^{\text{so}}, \epsilon, \eta) < \log \ell$, and

the supremum in (2.8) attained at a single point in the boundary ∂A reached by the corresponding level surface of $H(y)$.

If TM \mathbf{p}^{so} has $\mathbf{p}_{ij}^{\text{so}} = \mathbf{p}_j^{\text{so}}$ (an IID source), (2.9) is simplified. Introduce vector $\mathbf{p}^{\text{so}} = (\mathbf{p}_j^{\text{so}})$: here the IER $h = -\sum_i \mathbf{p}_i^{\text{so}} \log \mathbf{p}_i^{\text{so}}$, and $\gamma = \gamma(\mathbf{p}^{\text{so}}, \epsilon, \eta)$ is given by

$$\gamma = \sup \left[H(y) : y \in D \right], \quad (2.11)$$

where polyhedron $D = D(\mathbf{p}^{\text{so}}, \epsilon, \eta) \subset \mathbb{S}_\ell$:

$$D = \left\{ y = (y_j) \in \mathbb{S}_\ell : \sum y_i \varphi(i) \geq \eta \quad \text{and} \quad -\sum y_i \log \mathbf{p}_i^{\text{so}} \leq h + \epsilon \right\}. \quad (2.12)$$

Let us summarize. For an additive WF $\phi_n(\mathbf{x}) = \sum \varphi(x_i)$ the following result emerges:

Theorem 2.2 *Assume the source probabilities p_n^{so} are generated by an irreducible and aperiodic, stationary DTMC with an alphabet $\mathcal{X} = \{0, \dots, \ell - 1\}$, TM $\mathbf{p}^{\text{so}} = (\mathbf{p}_{ij}^{\text{so}})$ and equilibrium distribution $\pi^{\text{so}} = (\pi_i^{\text{so}})$. Set $h = -\sum_{i,j} \pi_i^{\text{so}} \mathbf{p}_{ij}^{\text{so}} \log \mathbf{p}_{ij}^{\text{so}}$. When selecting strings $\mathbf{x} \in \mathcal{X}^n$ with*

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi(x_i) \geq \eta \quad \text{and} \quad -\frac{1}{n-1} \log p_n^{\text{so}}(\mathbf{x}) \leq h + \epsilon, \quad (2.13)$$

the number b_n of selected strings satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log b_n = \gamma(\mathbf{p}^{\text{so}}, \epsilon, \eta) \quad (2.14)$$

where $\gamma(\mathbf{p}^{\text{so}}, \epsilon, \eta)$ is given by (2.8). For an IID source, with $\mathbf{p}_{ij}^{\text{so}} = \mathbf{p}_j^{\text{so}}$, one uses (2.11) with $h = -\sum_{j=1}^{\ell} \mathbf{p}_j^{\text{so}} \log \mathbf{p}_j^{\text{so}}$.

2.3

For completeness, we state an assertion for an WF $\phi_n(\mathbf{x}_0^{n-1}) = \sum_{i=0}^{n-k} \varphi(\mathbf{x}_i^{i+k-1})$ (when the summand WF φ takes into account k previous digits produced by the source) where $\mathbf{x}_i^{i+k-1} = (x_i, \dots, x_{i+k-1})$. Here we select strings $\mathbf{x}_0^{n-1} \in \mathcal{X}^n$ with

$$\frac{1}{n-k} \sum_{i=0}^{n-k} \varphi(\mathbf{x}_i^{i+k-1}) \geq \eta, \quad \frac{-1}{n-k} \log p_n^{\text{so}}(\mathbf{x}_0^{n-1}) \leq h + \epsilon \quad (2.15)$$

where h is as in (1.2). With $\mathcal{X} = \{0, \dots, \ell - 1\}$, assume that p_n^{so} are generated by a DTMC of order k , with state space \mathcal{X}^k , k -step transition probabilities $\mathbf{p}_{\mathbf{u}, \mathbf{u}'}^{\text{so}}$, $\mathbf{u}, \mathbf{u}' \in \mathcal{X}^k$, irreducible and aperiodic. Let $\pi_{\mathbf{u}}^{\text{so}}$ stand for the equilibrium probabilities and set

$$h = -\frac{1}{k} \sum_{\mathbf{u}, \mathbf{u}' \in \mathcal{X}^k} \pi_{\mathbf{u}}^{\text{so}} \mathbf{p}_{\mathbf{u}, \mathbf{u}'}^{\text{so}} \log \mathbf{p}_{\mathbf{u}, \mathbf{u}'}^{\text{so}}.$$

Theorem 2.3 *Adopt the above assumption. Similarly to (2.13), (2.14), the number b_n of selected strings satisfies*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log b_n = \gamma = \gamma(\epsilon, \eta). \quad (2.16)$$

Here γ is as follows: for $y = (y_i) \in \mathbb{S}_\ell$, set $H(y) = - \sum_{i \in \mathcal{X}} y_i \log y_i$, as in (2.7). Then

$$\gamma = \inf \left[H(y) : y \in B_{\ell,k}(\epsilon, \eta) \right], \quad (2.17)$$

with $B_{\ell,k} = B_{\ell,k}(\epsilon, \eta) \subset \mathbb{S}_\ell$:

$$B_{\ell,k} = \left\{ y = (y_j) : \exists \text{ a map } \mathbf{u} = (u_1, \dots, u_k) \in \mathcal{X}^k \mapsto \zeta(\mathbf{u}) \geq 0 \text{ such that} \right. \\ \left. \sum_{\mathbf{u}} \zeta(\mathbf{u}) \mathbf{1}(u_1 = j) = y_j \ \forall j \in \mathcal{X} \text{ and } \sum_{\mathbf{v}} \zeta(\mathbf{v}) \varphi(\mathbf{v}) \geq \eta, \right. \\ \left. -\frac{1}{k} \sum_{\mathbf{v}, \mathbf{v}'} \zeta(\mathbf{v}) \log p_{\mathbf{v}, \mathbf{v}'}^{\text{so}} \leq h + \epsilon \right\}. \quad (2.18)$$

For instance, take $k = 2$ (i.e., the source process is a DTMC of order two, and we work with $\varphi(i, j)$, $i, j \in \mathcal{X}$). Then

$$B_{\ell,2}(\epsilon, \eta) = \left\{ y = (y_i) \in \mathbb{S}_\ell : \exists \text{ a vector } z = (z_{ij}) \in \mathbb{R}_+^{\ell^2} \text{ such that } \sum_j z_{ij} = y_i, \text{ and} \right. \\ \left. \sum_{i,j} z_{ij} \varphi(i, j) \geq \eta, \quad \frac{1}{2} \sum_{i,j,k,l} z_{ij} \log p_{ij,kl}^{\text{so}} \leq h + \epsilon \right\}.$$

Remark 2.4 The bulk of the above analysis does not rely upon the particular form of the two-digit WF $(i, j) \in \mathcal{X} \times \mathcal{X} \mapsto -\log p_{ij}^{\text{st}}$ related to the information rate of a string. The choice of this WF (and of the upper bound $\sum T_{ij}^{(n)} \log p_{ij}^{\text{so}} \leq h + \epsilon$ in (2.2)) was made in order to connect with the Shannon NCT. In fact, the results stand up for any choice of a function $(i, j) \mapsto \varphi_2(i, j)$. However, selecting \mathcal{B}_n with $\frac{1}{n} \log p_n^{\text{so}}(\mathcal{B}_n) \leq \sigma < 0$ would lead to a further reduction of the memory volume needed to store set \mathcal{B}_n .

2.4 Examples

A. Let $\mathcal{X} = \{0, 1\}$ with $\ell = 2$ (a binary alphabet). Assuming that distributions p_n^{st} are generated by a DTMC, write the TM \mathbf{P}^{st} in the form

$$\mathbf{P}^{\text{st}} = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix}, \quad \text{with } \pi_0^{\text{st}} = \frac{\beta}{\alpha + \beta}, \quad \pi_1^{\text{st}} = \frac{\alpha}{\alpha + \beta},$$

where $\alpha, \beta \in (0, 1)$.

The analysis of maximization in (2.3) for $\ell = 2$ and given $y = (y_0, y_1) \in \mathbb{S}_2$, with $0 < y_0, y_1 < 1$, can be done in a straightforward (although tedious) manner. Recall: we want to find the maximum, in $0 < \mathbf{u} < 1$, of the expression

$$y_0 \ln \frac{\mathbf{u}}{(1 - \alpha)\mathbf{u} + \alpha(1 - \mathbf{u})} + y_1 \ln \frac{1 - \mathbf{u}}{\beta\mathbf{u} + (1 - \beta)(1 - \mathbf{u})} \\ = -y_0 \ln(1 - \alpha + \alpha\mathbf{w}) - y_1 \ln \left(1 - \beta + \frac{\beta}{\mathbf{w}} \right), \quad (2.19)$$

with $\mathbf{w} = \frac{1-u}{u} \in (0, \infty)$. It is convenient to maximize in \mathbf{w} . To this end, we solve

$$0 = \frac{\partial}{\partial \mathbf{w}} \left[-y_0 \ln(1 - \alpha + \alpha \mathbf{w}) - y_1 \ln \left(1 - \beta + \frac{\beta}{\mathbf{w}} \right) \right]$$

which is equivalent to the quadratic equation

$$y_0 \alpha (1 - \beta) \mathbf{w}^2 + \alpha \beta (y_0 - y_1) \mathbf{w} - y_1 \beta (1 - \alpha) = 0.$$

A solution $\mathbf{w} = K(y)$ has been identified in [8]:

$$K(y) = \frac{1}{2\alpha(1-\beta)y_0} \times \left[-\alpha\beta(y_0 - y_1) + \sqrt{(\alpha\beta(y_0 - y_1))^2 + 4\alpha\beta(1-\alpha)(1-\beta)y_0y_1} \right]. \quad (2.20)$$

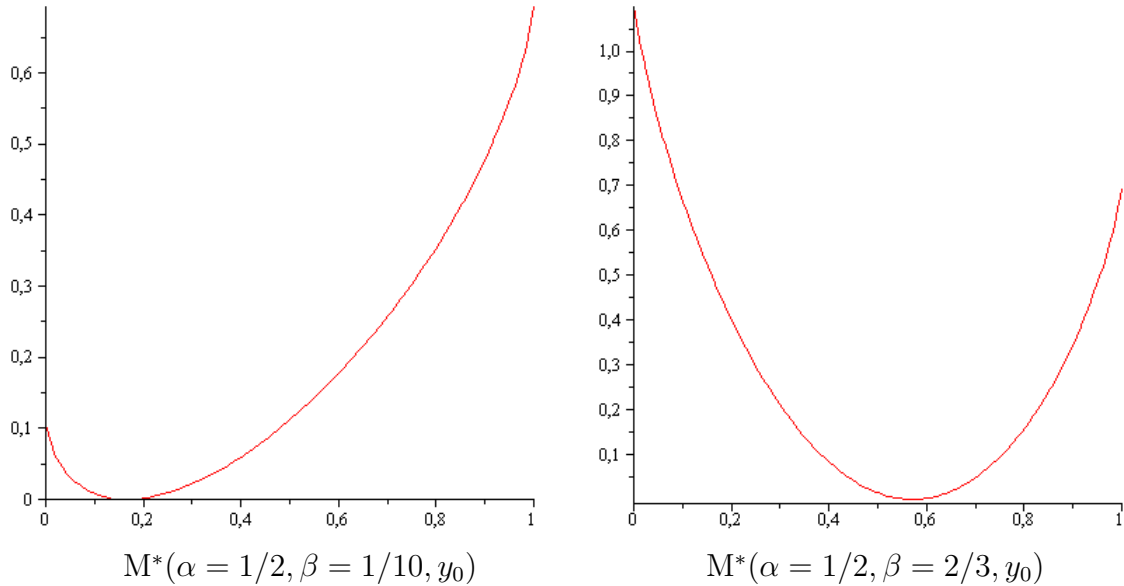
Then

$$M^*(y) = -y_0 \log(1 - \alpha + \alpha K) - y_1 \log(1 - \beta + \beta/K) \quad (2.21)$$

and

$$M^*(y) = \begin{cases} -\log(1 - \beta), & y_1 = 1, \\ -\log(1 - \alpha), & y_0 = 1. \end{cases}$$

It is true that $M^*(y) = 0$ if and only if $y_0 = \pi_0^{\text{st}}$, $y_1 = \pi_1^{\text{st}}$. Examples of graphs of function $y_0 \in (0, 1) \mapsto M^*(y)$ are given below. See also animations enclosed.



Accordingly, for $z = (z_{00}, z_{01}, z_{10}, z_{11}) \in \mathbb{S}_4$ with $z_{ij} \geq 0$ and $\sum z_{ij} = 1$, the value $\Pi^*(z)$ is given as follows. Set: $z^* = z_{00} + z_{01}$, $1 - z^* = z_{10} + z_{11}$ and $y^* = (z, 1 - z) \in \mathbb{S}_2$. Then

$$\Pi^*(z) = M^*(y^*). \quad (2.22)$$

B. Still with $\mathcal{X} = \{0, 1\}$ take $p_n^{\text{st}} = p_n^{\text{eq}}$. Suppose the source distributions p_n^{so} are generated by a DTMC with a transition matrix $\mathbf{P}^{\text{so}} = \begin{pmatrix} \mathbf{p}_{00}^{\text{so}} & \mathbf{p}_{01}^{\text{so}} \\ \mathbf{p}_{10}^{\text{so}} & \mathbf{p}_{11}^{\text{so}} \end{pmatrix}$, with $\pi_0^{\text{so}} = \frac{\mathbf{p}_{10}^{\text{so}}}{\mathbf{p}_{01}^{\text{so}} + \mathbf{p}_{10}^{\text{so}}}$, $\pi_1^{\text{so}} = \frac{\mathbf{p}_{01}^{\text{so}}}{\mathbf{p}_{01}^{\text{so}} + \mathbf{p}_{10}^{\text{so}}}$, the value $\gamma(\mathbf{P}^{\text{so}}, \epsilon, \eta)$ from (2.8) equals

$$\gamma(\mathbf{P}^{\text{so}}, \epsilon, \eta) = \sup \left[H(y) : y = (y_0, y_1) \in \mathbb{S}_2, y_1 \in A_2 \right] \quad (2.23)$$

where interval $A_2 = A_2(\mathbf{P}^{\text{so}}, \epsilon, \eta) \subseteq [0, 1]$ is given by

$$A_2 = \left\{ 0 \leq u \leq 1 : (1-u)\varphi(0) + u\varphi(1) \geq \eta, \exists z^{(j)} = (z_{j1}, z_{j2}) \in \mathbb{R}_+^2, j = 0, 1, \right. \\ \left. \text{such that } z_{00} + z_{01} = 1 - u, z_{10} + z_{11} = u, - \sum_{i,j=0}^1 z_{ij} \log \mathbf{p}_{ij}^{\text{so}} \leq h + \epsilon \right\}. \quad (2.24)$$

Here $H(y) = -y_0 \log y_0 - y_1 \log y_1$ and $h = - \sum_{i,j=0}^1 \pi_i^{\text{so}} \mathbf{p}_{ij}^{\text{so}} \log p_{ij}^{\text{so}}$.

Further, assuming $\mathbf{p}_{12}^{\text{so}} + \mathbf{p}_{21}^{\text{so}} = 1$, the above matrix \mathbf{P}^{so} has a repeated row $\mathbf{p} = (1 - \mathbf{p}, \mathbf{p})$ where $0 < \mathbf{p} < 1$. It yields an IID source, and the formula (2.23) for γ simplifies. We write $h = -(1 - \mathbf{p}) \log (1 - \mathbf{p}) - \mathbf{p} \log \mathbf{p}$, and

$$\gamma(\mathbf{p}; \epsilon, \eta) = \sup \left[H(y) : y = (y_0, y_1) \in \mathbb{S}_2, y_1 \in D_2 \right]. \quad (2.25)$$

Here interval $D_2 = D_2(\mathbf{P}^{\text{so}}, \epsilon, \eta) \subseteq [0, 1]$ is given by

$$D_2 = \left\{ 0 \leq u \leq 1 : (1-u)\varphi(0) + u\varphi(1) \geq \eta, \right. \\ \left. -(1-u) \log (1 - \mathbf{p}) - u \log \mathbf{p} \leq h + \epsilon \right\}. \quad (2.26)$$

We reiterate: the maxima in (2.23) and (2.25) are attained either at $u = 1/2$ – when $1/2 \in D_2(\epsilon, \eta)$, or at the nearest endpoint.

2.5

Next, we are going to (quickly) discuss multiplicative WFs $\phi_n(\mathbf{x}) = \prod_{i=0}^{n-1} \psi(x_i)$. Assuming that function ψ is strictly positive, consider selecting strings with $\phi_n(\mathbf{x}_0^{n-1}) \geq e^{n\eta}$. Passing to the logarithms yields

Theorem 2.5 *Under the assumptions of Theorem 2.2, select strings $\mathbf{x} \in \mathcal{C}^n$ where*

$$\mathcal{C}_n = \left\{ \mathbf{x} : \sum_i U_i^{(n)} \log \psi(i) \geq \eta, - \sum_{i,j} T_{ij}^{(n)} \log \mathbf{p}_{ij}^{\text{so}} \leq h + \epsilon \right\}. \quad (2.27)$$

Then, with $c_n = \#\mathcal{C}_n$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log c_n = \iota(\mathbf{P}^{\text{so}}, \epsilon, \eta). \quad (2.28)$$

Here $\iota(\mathbf{P}^{\text{so}}, \epsilon, \eta)$ is given by (2.8) with φ replaced by $\log \psi$.

Various generalizations can be achieved by following the same line of argument as for additive WFs.

3 The case of a general Markov source

When the alphabet set is large (or continuous), one can use a general theory where the source output is represented by a sequence of points in a space \mathcal{X} with some structure. Such a situation is typical when one stores analogous data. In particular, the volume in \mathcal{X} may be represented by a given measure ν with $V = \nu(\mathcal{X}) < \infty$. As above, the volume in \mathcal{X}^n may be associated with the product-measure ν^n or have a more involved form. Our aim here is similar: to assess the amount of volume needed to store valuable strings $\mathbf{x} = (x_0, \dots, x_{n-1}) \in \mathcal{X}^n$. A normalized product-volume $\frac{\nu^n}{V^n}$ yields a probability measure, an analog of p_n^{eq} ; asymptotic analysis of the volume is reduced to that of p_n^{eq} . More generally, as in Sect 2, we discuss the case where the volume is of the form $V_n p_n^{\text{st}}(\mathbf{x})$, assuming that p_n^{st} as well as the source distribution p_n^{so} are generated by DTMCs with state space \mathcal{X} . Transition matrices \mathbf{P}^{st} and \mathbf{P}^{so} are replaced with transition functions $\{\mathbf{P}^{\text{st}}(x, A)\}$ and $\{\mathbf{P}^{\text{so}}(x, A)\}$, $x \in \mathcal{X}$, $A \subseteq \mathcal{X}$; standard measurability conditions apply by default.

3.1

From now on \mathcal{X} is a Polish space with a chosen metric, $C_b(\mathcal{X})$ is the space of continuous bounded (real) functions $f : \mathcal{X} \rightarrow \mathbb{R}$ with the sup-norm and $\mathcal{P}(\mathcal{X})$ the space of Radon probability measures ν on \mathcal{X} with the Lévy–Prokhorov metric. (Then $\mathcal{P}(\mathcal{X})$ is a Polish space.) In a similar manner, consider the space $\mathcal{P}(\mathcal{X} \times \mathcal{X})$. Let us fix a non-negative finite Radon measure ν on \mathcal{X} and designate $\mathcal{P}_\nu = \mathcal{P}_\nu(\mathcal{X})$ to be the set of measures absolutely continuous relative to ν . Next, let $\mathcal{P}_{\nu \times \nu} = \mathcal{P}_{\nu \times \nu}(\mathcal{X} \times \mathcal{X})$ designate the set of measures absolutely continuous relative to $\nu \times \nu$.

We suppose that measures $\mathbf{P}^\bullet(x, \cdot)$ are absolutely continuous relative to ν and work with the corresponding transition densities $\mathbf{p}^\bullet(x, x') = \frac{\mathbf{P}^\bullet(x, dx')}{\nu(dx')}$, $x, x' \in \mathcal{X}$. For $\mathbf{p}^{\text{st}}(x, x')$ we also adopt Assumption (U) from [10], Ch. 6.3. (There exists a host of weaker conditions; see Assumptions (H-1), (H-2) from [10], Ch. 6.3 and from [11], Ch. 5.4, leading to more involved formulas.) For $\mathbf{p}^{\text{so}}(x, x')$ we assume ergodicity under a unique equilibrium distributions $\pi^{\text{so}} \in \mathcal{P}_\nu$ and suppose that the integral giving the IER converge absolutely:

$$h = h^{\text{so}} = \int_{\mathcal{X} \times \mathcal{X}} \log \mathbf{p}^{\text{so}}(x, x') \pi^{\text{so}}(dx) \mathbf{P}^{\text{so}}(x, dx'). \quad (3.1)$$

As was said, p_n^{st} stands for the probability measure on \mathcal{X}^n generated by the DTMC with transition density $\mathbf{p}^{\text{st}}(x, y)$ under a given initial distribution λ .

Let $\mathbf{x} \in \mathcal{X}^n$. Following (2.1), consider empirical measures $\mathbf{U}^{(n)} = \mathbf{U}^{(n)}(\mathbf{x}) \in \mathcal{P}(\mathcal{X})$ and $\mathbf{T}^{(n)} = \mathbf{T}^{(n)}(\mathbf{x}) \in \mathcal{P}(\mathcal{X} \times \mathcal{X})$:

$$\mathbf{U}^{(n)} = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{x_j}, \quad \mathbf{T}^{(n)} = \frac{1}{n-1} \sum_{j=0}^{n-2} \delta_{x_j, x_{j+1}}. \quad (3.2)$$

Here δ stands for the Dirac mass.

According to standard LDP results, $\mathbf{U}^{(n)}$ and $\mathbf{T}^{(n)}$ satisfy the full LDP (in $\mathcal{P}_\nu \times \mathcal{P}_{\nu \times \nu}$) with good convex LDR functions $M^*(v)$ and $\Pi^*(\tau)$, $v \in \mathcal{P}_\nu$, $\tau \in \mathcal{P}_{\nu \times \nu}$. See [10], Ch 6.3, particularly, Theorem 6.3.8. Moreover, M^* and Π^* can be specified as follows.

- (i) When $v \in \mathcal{P}(\mathcal{X}) \setminus \mathcal{P}_\nu$ or $\tau \in \mathcal{P}(\mathcal{X} \times \mathcal{X}) \setminus \mathcal{P}_{\nu \times \nu}$, we have $M^*(v) = \Pi^*(\tau) = \infty$.
- (ii) For $v \in \mathcal{P}_\nu$ and $\tau \in \mathcal{P}_{\nu \times \nu}$,

$$\begin{aligned} M^*(v) &= \sup \left[\int_{\mathcal{X}} \log \frac{m(x)}{\mathbf{P}^{\text{st}} m(x)} v(dx) : m \in C_b(\mathcal{X}), m \geq 1 \right], \\ \Pi^*(\tau) &= \sup \left[\int_{\mathcal{X}} \log \frac{m(x)}{\mathbf{P}^{\text{st}} m(x)} \tau(\mathcal{X} \times dx) : m \in C_b(\mathcal{X}), m \geq 1 \right] \end{aligned} \quad (3.3)$$

where $\mathbf{P}^{\text{st}} m(x) = \int_{\mathcal{X}} \mathbf{p}^{\text{st}}(x, y) m(y) \nu(dy)$.

Cf. [11], Ch. 4.1, and [10], Ch. 6.5 (detailed references have been given at the beginning of Sect 2.2).

3.2

For an additive WF $\phi_n(\mathbf{x}_0^{n-1}) = \sum_{j=0}^{n-1} \varphi(x_j)$ we assume that function $\varphi : \mathcal{X} \rightarrow \mathbb{R}$ is continuous. We want to select strings from $\mathcal{B}_n = \mathcal{B}_n(\epsilon, \eta)$ where

$$\mathcal{B}_n = \left\{ \mathbf{x} \in \mathcal{X}^n : \frac{1}{n} \sum_{i=0}^{n-1} \varphi(x_i) \geq \eta \text{ and } -\frac{1}{n-1} \sum_{i=0}^{n-2} \log \mathbf{p}^{\text{so}}(x_i, x_{i+1}) \leq h + \epsilon \right\} \quad (3.4)$$

and h is given in (3.1). Equivalently,

$$\begin{aligned} \mathcal{B}_n &= \left\{ \mathbf{x} : \int_{\mathcal{X}} \varphi(x) \mathbf{U}^{(n)}(dx) \geq \eta \text{ and } \right. \\ &\quad \left. - \int_{\mathcal{X} \times \mathcal{X}} \log \mathbf{p}^{\text{so}}(x, x') \mathbf{T}^{(n)}(dx \times dx') \leq h + \epsilon \right\}. \end{aligned} \quad (3.5)$$

Now, with $\Pi^*(\tau) = \Pi^*(\mathbf{P}^{\text{st}}, \tau)$ as in (3.3), set:

$$\kappa(\epsilon, \eta) = \kappa(\epsilon, \eta, \mathbf{P}^{\text{st}}, \mathbf{P}^{\text{so}}) = -\inf \left[\Pi^*(\tau) : \tau \in B \right], \quad (3.6)$$

where set $B = B(\mathbf{P}^{\text{so}}, \epsilon, \eta) \subset \mathcal{P}_{\nu \times \nu}$ is given by

$$\begin{aligned} B &= \left\{ \tau : \int_{\mathcal{X} \times \mathcal{X}} \varphi(x') \tau(dx \times dx') \geq \eta \text{ and } \right. \\ &\quad \left. - \int_{\mathcal{X} \times \mathcal{X}} \log \mathbf{p}^{\text{so}}(x, x') \tau(dx \times dx') \leq h + \epsilon \right\}. \end{aligned} \quad (3.7)$$

Theorem 3.1 *Under the above assumptions, for all $\epsilon, \eta > 0$, the following relation holds true:*

$$\kappa(\epsilon, \eta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log p_n^{\text{st}}(\mathcal{B}_n). \quad (3.8)$$

In the case of volume ν^n in \mathcal{X}^n , the above formulas simplify. Let us set:

$$\gamma(\epsilon, \eta) = \gamma(\mathbf{P}^{\text{so}}, \epsilon, \eta) = \inf \left[H(v) : v \in A \right]. \quad (3.9)$$

Here set $A = A(\mathbf{P}^{\text{so}}, \epsilon, \eta) \subset \mathcal{P}_\nu$ is given by

$$A = \left\{ v : \int_{\mathcal{X}} \varphi(x) v(dx) \geq \eta \text{ and } \exists \text{ a measure } \tau \in \mathcal{P}_{\nu \times \nu} \text{ with} \right. \\ \left. \tau(\mathcal{X} \times dx') = v(dx') \text{ and } - \int_{\mathcal{X} \times \mathcal{X}} \log \mathbf{p}^{\text{so}}(x, x') \tau(dx \times dx') \leq h + \epsilon \right\} \quad (3.10)$$

and for $\mu \in \mathcal{P}_\nu$ with $m(x) = \frac{\mu(dx)}{\nu(dx)}$,

$$H(\mu) = - \int_{\mathcal{X}} m(x) \log m(x) \nu(dx). \quad (3.11)$$

If $\mathbf{p}^{\text{so}}(x, x') = \mathbf{p}^{\text{so}}(x')$ (i.e., in the case of IID source outputs), the formula for γ is further streamlined. It is expressed in terms of density $\mathbf{p}^{\text{so}}(x')$: here the entropy rate $h = - \int_{\mathcal{X}} \mathbf{p}^{\text{so}}(x') \log \mathbf{p}^{\text{so}}(x') \nu(dx')$, and

$$\gamma(\epsilon, \eta) = \inf \left[H(v) : v \in D \right], \quad (3.12)$$

with $D = D(\mathbf{P}^{\text{so}}, \epsilon, \eta) \subset \mathcal{P}_\nu$:

$$D = \left\{ v : \int_{\mathcal{X}} \varphi(x) v(dx) \geq \eta, - \int_{\mathcal{X}} \log \mathbf{p}^{\text{so}}(x) v(dx) \leq h + \epsilon \right\}. \quad (3.13)$$

The above construction leads to following result.

Theorem 3.2 *Let the source process be an ergodic DTMC with states $x, x' \in \mathcal{X}$, transition densities $\mathbf{P}^{\text{so}} = \{\mathbf{p}^{\text{so}}(x, x')\}$ and equilibrium density $\pi^{\text{so}}(x)$. Let h stand for the IER. Consider the volume $v_n = \nu^n(\mathcal{B}_n)$ of set $\mathcal{B}_n \subset \mathcal{X}^n$ as in (3.4). Then*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log v_n = \gamma(\mathbf{P}^{\text{so}}, \epsilon, \eta) \quad (3.14)$$

where $\gamma(\epsilon, \eta)$ is given by (3.9). For an IID source, with $\mathbf{p}^{\text{so}}(x, x') = \mathbf{p}^{\text{so}}(x')$, one uses Eqn (3.12).

Remark 3.3 The entropy functional $\mu \in \mathcal{P}_\nu \mapsto H(\mu)$ in (3.11) is concave, and set D in (3.12) is convex. It is tempting to conjecture that if D does not contain the probability measure ν/V (the global maximizer of H in \mathcal{P}_ν) then the maximum of $H(v)$ over D is attained at a unique point lying in a (suitably defined) boundary ∂D . This direction needs further exploring; many aspects of convexity and related topics of optimisation are discussed in [21]–[23].

3.3

For a multiplicative WFs $\phi_n(\mathbf{x}) = \prod_{i=0}^{n-1} \psi(x_i)$ with strictly positive one-digit factor $\psi(x)$, we obtain the following assertion:

Theorem 3.4 *Under the assumptions of Theorem 2.2, select strings $\mathbf{x} \in \mathcal{C}^n$ where $\mathcal{C}_n \subset \mathcal{X}^n$ is as in (2.27). Then, for $w_n = \nu^n(\mathcal{C}_n)$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log w_n = \iota(\mathbf{P}^{\text{so}}, \epsilon, \eta). \quad (3.15)$$

Here $\iota(\mathbf{P}^{\text{so}}, \epsilon, \eta)$ is given by (3.9) and (3.10) with φ replaced by $\log \psi$.

Concluding remarks

The paper discusses the problem of storing ‘valuable’ data (digital or analogous) selected on the basis of the rate of a utility/weight function. The storage space is treated as an expensive commodity that should be provided and organized in an efficient manner. The issue of reducing and organising storage space is addressed from a probabilistic point of view which is an extension of the Shannon data-compression principle (the Shannon Noiseless coding theorem). More precisely, the storage volume is assessed via the theory of large deviations. The emerging optimization problem is highlighted and explained through examples.

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